ON H″-CLOSED SETS IN TOPOLOGICAL SPACES

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Abstract: **In this article, we introduce a new class of closed sets in topological spaces namely, H″-closed and we prove every subset of the digital line is H″-closed.**

Keywords: H″-closed sts, H″-open sets Λ-sets and Λ_r-sets.

1 INTRODUCTION

J. Jeyanthi et.al the introduced Λ_r-closed, Λ_r-continuous and Caldas et.al the introduced λ-closed and λ-continuous. Levin introduced generalized closed sets developed by more generalized sets.

In this paper, we introduce a new class of closed sets in topological spaces namely, H″-closed and we prove every subset of the digital line is H″-closed.

Through out paper obtained in the Topological space (X,τ) (resp. (X,σ) and (X,η)) is denoted by TS X (resp. TS Y and TS Z).

For a subset C of a TS X, int(C), cl(C) denoted the interior, closure of C respectively. And λ symbol use this thesis A.

For so many author introduced various definitions

Definition 1.1. [5] *The collection of all λ-closed(resp. λ-open) subsets of X will be denoted by $\lambda C(X)$ (resp. $\lambda O(X)$). We set*
$\lambda C(X,x) = \{U : x \in O \in \lambda C(X,\tau)\}$ $\lambda O(X,x) = \{U : x \in O \in \lambda O(X,\tau)\}$

Definition 1.2. [8] *Let B be a subset of a TS X. We define subsets B^Λ and B^V as follows:*

$$B^\Lambda = \cap\{U/U \supseteq B, U \in \tau\}, and$$

$$B^V = \cup\{F/F \subseteq B, X - F \in \tau\}$$

A subset B of (X,τ) is a Λ-set (resp. V-set) if $B = B^\Lambda$ (resp. $B = B^V$).

Definition 1.3. *A subset C of a TS X is called a*

(1) *g-closed set* [7] *if Cl(C) ⊆ U whenever C ⊆ U and U is open.*
The complement of g-closed set is g-open set.

(2) *a Λ_g-closed set* [4] *(resp.Λ-g-closed* [5], *gΛ-closed) if $Cl(C) \subseteq U$ (resp. $Cl_\lambda(C) \subseteq U$, $Cl_\lambda(C) \subseteq U$) whenever $C \subseteq U$ and U is λ-open (resp. U is λ-open, U is open).*

Lemma 1.4. [2] *Let C be a subset of a TS X. Then we have the next:*

(1) *If $C \subset X$ then $C \subset \lambda\, ker(C)$.*
(2) *If $C,B \subset X$ then $C \subset B$ implies $\lambda Ker(C) \subset \lambda Ker(B)$.*
(3) *$\lambda Ker(\lambda Ker(C)) = \lambda Ker(C)$.*

Proposition 1.5. [8]

(1) *The subsets φ and X are Λ-sets.*
(2) *Every union of Λ-sets is a Λ-set.*
(3) *Every intersection of Λ-sets is a Λ-set.*
(4) *A subset B is a Λ-set if and only if the complement of B is a V-set.*

Lemma 1.6. [9] *Every Λ_r-set is Λ-set.*

H″-sets and K″-sets

This section contains a new class of sets, called H″-sets in TS and investigate certain basic properties of H″-sets.

Definition 2.1. *Let S be a subset of a TS X, then we define a $SS'' = \cap\{Q/Q \supset S, Q \in AO(X,\tau)\}$.*

Lemma 2.2. [5] *Let C, B and $C_i (i \in I)$ be a subset of a TS X. The following properties hold:*

(1) $C \subset Acl(C) \subset cl(C)$.
(2) $C \subset B \Longrightarrow Acl(C) \subset Acl(B)$.
(3) *C is* A-*closed* $\Longleftrightarrow C = Acl(C)$.
(4) $Acl(Acl(C)) = Acl(C)$. (5) *If C_i is* A-*closed for each $i \in I$ then $\bigcap_{i\in I} C_i$ is* A-*closed.*
(6) *If C_i is* A-*open for each $i \in I$ then $\bigcup_{i\in I} C_i$ is* A-*open.*

Lemma 2.3. *For the subsets C, B and $C_i (i \in I)$ of a TS X the following hold.*

(1) *If $C \in AO(X,\tau)$ then $C = CS''$.*

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(2) $(\cup_{i\in I} C_i)S'' = \cup_{i\in I} C_iS''$.

(3) $(\cap_{i\in I} C_i)S \subset \cap_{i\in I} C_iS$.

Proof. (1) By Definition 2.1 and since $C \in AO(X,\tau)$, we have $CS'' \subseteq C$. By Lemma 1.4(1), we have that $C = CS''$.

(2) Assume that there exists a point $x \in X$ such that $x \notin \cup_{i\in I} C_i^S$. Then by Definition 2.1, there exists subsets $W_i \in AO(X,\tau)$, for all $i \in I$, such that $x \notin W_i$, $C_i \subset W_i$. Let $W = \cup_{i\in I} P_i$. Then we have that $x \notin \cup_{i\in I} W_i$, $\cup_{i\in I} C_i \subseteq W$ and $W \in AO(X,\tau)$. This implies that $x \notin (\cup_{i\in I} C_i)^{\ddot{S}}$. Thus $(\cup_{i\in I} C_i)^{\ddot{S}} \subset \cup_{i\in I} C_i^{\ddot{S}}$.

Conversely, Assume that there exists a point x such that $x \notin (\cup_{i\in I} C_i)^{\ddot{S}}$. Then there exists a subset $W \in AO(X,\tau)$ such that $\cup C_i \subset W$ and $x \notin W$.

Thus, for each $i \in I$ we have $x \notin C_i^S$. This implies that $x \notin \cup_{i\in I}(C_i^{\ddot{S}})$. Thus, $\cup_{i\in I} C_i^{\ddot{S}} \subset (\cup_{i\in I} C_i)^{\ddot{S}}$.

(3) Suppose that there exist a point x such that $x \notin \cap_{i\in I} C_iS''$. Then there exists $i_0 \in I$ such that $x \notin (C_{i0})S$ and there exists A-open set P such that $x \notin P$ and $C_{i0} \subset P$. We have $\cap_{i\in I} C_i \subset C_{i0} \subset P$ and $x \notin P$. Therefore, $x \notin (\cap_{i\in I} C_i)S''$.

This shows that $(\cap_{i\in I} C_i)^{\ddot{S}} \subset \cap_{i\in I} C_i^{\ddot{S}}$.

Remark 2.4. *The converse of above Theorem 2.3 (1) is not true and the equality of (3) is not always true in general.*

Example 2.5. *Let* $X = \{1,2,3,4\}$ *with* $\tau = \{X, \varphi, \{1, 3\}, \{1, 3, 4\}\}$.
By Definition 2.1, $(\{4\})S'' = \{4\}$, *the set* $\{4\}$ *is not* A-*open.*

Definition 2.6. *A subset C of a TS X is called* S″-*set if* $C = CS''$.
The family of all H″*-sets of TS X is denoted by* $\tau S''(X)$*(or simply* $\tau S''$*).*

Proposition 2.7. *In a TS X, every* Λ*-set is* S″*-set.*
Proof. Follows from the fact that every open set is A-open.

Example 2.8. *Let* $X = \{1,2,3\}$ *and* $\tau=\{\varphi, X, \{1\}, \{1, b\}\}$*. Hence,* $\{2\}$ *is a* S″*-set but not a* Λ*-set.*

Definition 2.9. *Let Q be a subset of a TS X then we define the following:*
$QK'' = \cap\{P/P \subset Q, P \in AC(X,\tau)\}$.

Definition 2.10. *A subset C of a TS X is called* K″*-set if* $C = CK''$.
The family of all K″-sets of TS X is denoted by $\tau K''(X)$(or simply $\tau K''$).

Lemma 2.11. *For subsets C and* $C_i (i \in I)$ *of a TS X, the following hold.*

(1) *The subsets* φ *and X are* S″*-sets.*
(2) *If C is* A*-open then C is a* S″*-set.*
(3) *If* C_i *is a* S″*-set for each* $i \in I$ *then* $\cup_{i\in I} C_i$ *is a* S″*-set.*
(4) *If* C_i *is a* S″*-set for each* $i \in I$ *then* $\cap_{i\in I} C_i$ *is a* S″*-set.*

Proof. (1) Follows from Proposition 1.5(1) and Proposition 2.7.
(2) Follows from Lemma 2.3(1) and Definition 2.6.
(3) Let $C_i \in \tau S''$ for some $i \in I$, then we have, $\cup_{i\in I} C_i = \cup_{i\in I} C_i^{\ddot{S}}$ by Definition 2.6 $\cup C_i = \cup C_iS'' = (\cup C_i)S''$ by Lemma 2.3, $\cup C_i = \cup C_iS'' = (\cup C_i)S'' \supset \cup C_i$ by Lemma 1.4(1). Thus, we have $\cup_{i\in I} C_i = (\cup_{i\in I} C_i)S$. Therefore $\cup_{i\in I} C_i \in \tau S$.

(4) Let $C_i \in \tau S$ for each $i \in I$, then by Definition 2.6, Lemma 2.3 and 1.4(1) we have, $\cap_{i\in I} C_i = \cap_{i\in I} C_i^{\ddot{S}} \supset (\cap_{i\in I} C_i)^{\ddot{S}} \supset \cap_{i\in I} C_i$. Thus, we have $\cap_{i\in I} C_i = (\cap_{i\in I} C_i)^{\ddot{S}}$ and $\cap_{i\in I} C_i \in \tau^{\ddot{S}}$.

Theorem 2.12. *In a TS X, if* $AO(X,\tau) = AC(X,\tau)$ *then for any subset* $C \subset X$, $Acl(C) = CS''$.
Proof. Let $C \subset X$ and $C \in AO(X,\tau)$. By Lemma 2.3(1) we have, $CS'' = C$. By assumption, $C \in AC(X,\tau)$. Then, by Lemma 2.2(3) $Acl(C) = C$. Therefore $Acl(C) = CS''$.

Remark 2.13. *The converse of Theorem 2.12 is not true in general.*

Example 2.14. *Let* $X = \{1,2,3,4,5\}$ *and* $\tau=\{\varphi, X, \{3, 4\}, \{1, 3, 4\}\}$*. Let* $C = \{1\}$*. Then* $Acl(C) = \{1\}$ *and* $AS'' = \{1\}$ *but* $AO(X,\tau) \; 6= AC(X,\tau)$.

H″-closed sets

We introduce a new class of sets, called H″-closed sets in space and study

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their properties.

Definition 3.1. *A subset C of a TS X is called* H″-*closed if* $C = P \cap Q$ *where P is a* H″-*set and Q is a* A-*closed set.*

The complement of H″-*closed set is called* H″-*open set.*

Lemma 3.2. *Let a TS X. Then the following properties are valid:*

(1) φ *and X are* A-*closed and* A-*open in X.*
(2) A-*closed set in X is* H″-*closed in X.*
(3) φ *and X are* H″-*closed and* H″-*open in X.*

Proof. (1) Since φ can be written as $\varphi \cap X$, X is closed in TS X and by Proposition 1.5(1), φ is a Λ-set we have φ is a A-closed set in TS X. Since $X = X \cap X$, X is closed in TS X and by the Proposition 1.5, X is a Λ-set, X is A-closed in X. The complement of φ and X is X and φ respectively. Hence φ and X are A-open in X.

(2) Let A be a A-closed set in X. X is A-open. By Lemma 2.11 (2), X is a S″-set in X. Hence $C = C \cap X$ is a H″-closed set in X.

(3) By (1) and (2), we get φ and X are H″-closed. The complement of φ and X is X and φ respectively. Hence φ and X are H″-open in X.

Remark 3.3. *In general the converse of (2) of Lemma 3.2 is not true which is seen in the next Example.*

Example 3.4. *Let a TS X such that* $X = \{1,2,3,4\}$ *and* $\tau = \{\varphi, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}, X\}$. *Hence* $\{1,2,4\}$ *is* H″-*closed but not* A-*closed.*

Theorem 3.5. *In a TS X, every* Λ_r-*closed is* H″-*closed.*

Proof. Suppose $C = P \cap Q$ is Λ_r-closed, where P is a Λ_r-set and Q is closed. By Lemma 1.6, P is a Λ-set. Thus C is A-closed and hence C is H″-closed.

Example 3.6. *Let* $X = \{1,2,3,4\}$, $\tau = \{\varphi, X, \{1, b\}, \{1, 2, 3\}\}$. *Hence* $\{3\}$ *is not a* Λ_r-*closed set but it is a* H″-*closed set.*

Theorem 3.7. *In a TS X, every* (Λ,θ)-*closed is* (Λ,δ)-*closed.*

Proof. Follows from the fact that every θ-closed is δ-closed and θ-open is δ-open.

Example 3.8. *Let* $X = \{1,2,3,4\}$, $\tau=\{\varphi, X, \{1\}, \{2\}, \{1, 2\}\}$. *Hence* $\{1\}$ *is a* (Λ,δ)-*closed set but not a* (Λ,θ)-*closed set.*

Lemma 3.9. *In a TS X, every* Λ_θ-*set is* Λ-*set.*

Proof. Follows from the fact that every θ-open is open.

Example 3.10. *Let* $X = \{1,2,3\}$, $\tau=\{\varphi, X, \{1\}, \{1, 2\}\}$. *Hence* $\{1\}$ *is a* Λ-*set but not a* Λ_θ-*set.*

Theorem 3.11. *In a TS X, every* (Λ,θ)-*closed is* H″-*closed.*

Proof. Suppose $C = P \cap Q$ is Λ_θ-closed, where P is a Λ_θ-set and Q is θ-closed. By Lemma 3.9, L is a Λ-set. Since every θ-closed set is a closed set, Q is closed. Thus C is A-closed and hence C is H″-closed.

Example 3.12. *Let* $X = \{1,2,3,4\}$, $\tau=\{\varphi, X, \{1\}, \{2\}, \{1, 2\}\}$. *Hence* $\{1\}$ *is* H″-*closed but not* (Λ,θ)-*closed.*

Remark 3.13. *The concepts of closed sets and* (Λ,δ)-*closed sets are independent.*

Example 3.14.

(1) Let $X = \{1,2,3\}$, $\tau=\{\varphi, X, \{1\}, \{1, 2\}\}$. Hence $\{3\}$ is closed but not (Λ,δ)-closed.
(2) Let $X = \{1,2,3\}$, $\tau=\{\varphi, X, \{1\}, \{2\}, \{1, 2\}\}$. Here $\{3\}$ is (Λ,δ)-closed but not closed.

Lemma 3.15. *In a TS X, every* Λ_δ-*set is* Λ-*set.*

Proof. Follows from the fact every δ-open is open.

Example 3.16. *Let* $X = \{1,2,3\}$, $\tau=\{\varphi, X, \{1\}\}$. *Hence* $\{1\}$ *is a* Λ-*set but not a* Λ_δ-*set.*

Theorem 3.17. *In a TS X, every* (Λ,δ)-*closed is* H″-*closed.*

Proof. Suppose $C = P \cap Q$ is Λ_δ-closed, where P is a Λ_δ-set and Q is δ-closed. By Lemma 3.15, P is a Λ-set. Since every δ-closed set is a closed set, Q is closed. Thus C is A-closed and hence C is H″-closed.

Example 3.18. *Let* $X = \{1,2,3\}$, $\tau=\{\varphi, X, \{1\}, \{1, 2\}\}$. *Hence* $\{1\}$ *is* H″-*closed but not* (Λ,δ)-*closed.*

Theorem 3.19. *For a subset C of a TS X, the following properties are equivalent.*

(1) *C is* H″-*closed.*
(2) $C = P \cap \mathrm{A}cl(C)$, *where P is a* H″-*set.*
(3) $C = C^{S''} \cap \mathrm{A}cl(C)$.

Proof. (1) ⇒ (2) : Let C be H″-closed, then there exists a H″-set P and a A-closed such that $C = P \cap Q$. Since $C \subset Q$, We have $C \subset \mathrm{A}cl(C) \subset \mathrm{A}cl(Q) = Q$ and $C = P \cap Q \supset P \cap \mathrm{A}cl(C) \supset C$. Therefore we obtain $C = P \cap \mathrm{A}cl(C)$.

(2) ⇒ (3) : Let $C = P \cap \mathrm{A}cl(C)$, where P is a H″-set. Since $C \subset P$, we have $C^{S''} \subset P^{S''} = P$ and hence $C \subset C^{S''} \cap \mathrm{A}cl(C) \subset P \cap \mathrm{A}cl(C) = C$. Therefore, we obtain $C = C^{S''} \cap \mathrm{A}cl(C)$.

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(3) ⇒ (1) : By Lemma 1.4, CS'' is a S''-set and by Lemma 2.2(4), $Acl(C)$ is A-closed. By (3), $C = CS'' \cap Acl(C)$ and hence C is H''-closed.

Theorem 3.20. *If a set C is Λ-g-closed then $Acl(C)/C$ contains no non empty A-closed.*

Proof. Let G be a A-closed subset of $Acl(A) - C$. Now, we have $C \subseteq X - G$. Since C is Λ-g-closed, we have $Acl(C) \subseteq X - G$ (or) $G \subseteq X - (Acl(C))$. Thus $G \subseteq Acl(C) \cap (X - (Acl(C))) = \varphi$ and G is φ.

Remark 3.21. *The converse of Theorem 3.20 is not true as it can be seen by the next Example.*

Example 3.22. *Let $X = \{1,2,3,4\}$, $\tau = \{\varphi, X, \{1\}, \{1, b\}\}$. If $C = \{1,3\}$ then $Acl(C) - C = \{2,4\}$ does not contain nonempty A-closed set. But C is not Λ-g-closed.*

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